The dynamical systems and their fourier modes

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The Dynamical systems and their Fourier Modes

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Abstract

We have studied the behaviors of several dynamical systems described the differential difference equations as pattern translations. The trajectories, the Fourier components of their pattern and their phase portraits in the time difference and the space difference are shown.

§ 1. Introduction

The dynamical systems are defined by the time evolution equations. So they are deterministic and have decided future orbits for a given initial condition. However the future orbits of neighboring points are not always near each other, but can not be predictable in many nonlinear systems. This kind of unpredictability is called chaos. On the other hand the integrable nonlinear systems have stable soliton-like solutions. It may not necessarily be correct that chaos and integrability are antinomic concepts, because general unique definitions are not always clear for chaos and integrability. Since the complexity of solutions in three body systems was found by Poincare, many chaotic behaviors have been observed especially in computer simulations. We have showed chaotic behaviors of Volterra-type models and other systems with various initial conditions in our previous papers [1]. On the other hand the Fourier transfom is useful tool to investigate of periodic behaviors or randam process. In the present article, we are going to perform the Fourier transform of periodic and chaotic behaviors in several discrete dynamical systems. In the next section (§ 2), the types of dynamical systems are reviewed and initial values are recongnized as a pattern. In § 3, Fourier transform and power spectrum are described. In the last section (§ 4), the numerical simulations of several equations are shown.

§ 2. Dynamical Systems and Pattern Translations

There are three types in the dynamical systems [2] :
the first type is continuous dynamical system described in terms of differential equation

$$\frac{dX}{dt} = f(X) \tag{1}$$

the second type is discrete dynamical system described in terms of difference equation

$$X_{n+1} = f(X_n) \tag{2}$$

and the third type is symbol dynamical system described in terms of symbols and shift operator which operate on the sequence space.

We may add complex analytic dynamical system, in which complex analytical maps divide the plane two sets : stable set and chaotical Julia set [3].

However these types are not clearly separated, but they are often mixed. For example, difference equations are used to solve differential equations of continuous dynamics in computer simulation, and there are many finite difference methods for a differential equation [4]. We have studied the Volterra differential-difference equation in the previous papers [1]

$$\frac{df_n}{dt} = (f_{n+1} - f_{n-1}) f_n \tag{3}$$

where difference equation is used for space variable n and differential equation is used for time variable t. To solve this nonlinear differential-difference equation, we have used follwing Hirota's difference-difference equation [5]

$$f_n^{t+1}-f_n^t=\delta(f_n^t f_{n-1}^{t+1}-f_n^{t+1} f_{n+1}^t), \tag{4}$$

and difference equation by the simple finite difference method for time derivative

$$f_n^{t+1}-f_n^t=\delta(f_n^t f_{n+1}^t-f_n^t f_{n-1}^t). \tag{5}$$

The difference between these two difference equations is clear for the initial conditions with the exact solutions of Eq. (3) [1].

We started from the zero curvature condition for the integrable system [6]

$$\frac{\partial M}{\partial t}-\frac{\partial L}{\partial x}+[M,L]=0 \tag{6}$$

where the time and space variables are formally symmetric. The dynamical systems are defined as the evolution equations in time, so the symmetry is broken. Spreading in space may be recongnized as a pattern. We took the initial values as the initial condition, in a sense, a form of string at $t=0$. It may be said that we deal a so-called pattern and pattern tanslation by different types of mapping : dynamical systems, which are divided into linear or nonlinear and integrability or not.

§ 3. Fourier Transform

In general any function defined in the finite region is represented by the orthonormal set. One of the most popular system is Fourier series, and its extension to infinite region is Fourier transfom. This transformation supply us the world of the frequency space which is complement of the time-space world : representation in the frequency field for the structure of the function in the time field [7], and Fourier transform between space variable and wave number corespond to one in the wave-number field for the structure of the function in the space field. Fourier transfom of a function $f(t)$ is given by

$$F(w)=\int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-iwt}dt \tag{7}$$

and its inverse transformation is

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwt}dw \tag{8}$$

Where t is a time variable and ω is angular frequency. In Practice we must actually deal with finite discrete data, so we use Fourier series

$$F_l=\sum_{k=1}^{N-1}f_k e^{-it\Delta wk\Delta t}\Delta t \qquad l=0,\ 1,\ \cdots,\ N-1 \tag{9}$$

$$f_k=\sum_{l=1}^{N-1}F_l e^{-it\Delta wk\Delta t}\Delta w \qquad k=0,\ 1,\ \cdots,\ N-1 \tag{10}$$

where N is number of data, $f_k=f(k\Delta t)$, $F_l=F(l\Delta\omega)$, Δt is the time interval of data, and $\Delta\omega=2\pi/(N\Delta t)$.

The fast Fourier transfom (FFT) is convenient to treat massive data very quickly. However there are problems in this FFT that use of Eqs. (9) (10) instead of Eqs. (7) (8) coresponds to use periodically repeated data [8] and information in the high frequency region greater than Nyquist frequency ($1/2\,\Delta t$) is lost [9, 10]. On the other hand the power spectral density is defined as the absolute magnitude of the Fourier transform of auto-correlation function $C(\tau)$ [9]

$$C(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)f(t+\tau)\,dt \tag{11}$$

which is a kind of convolution integral. Since chaos is accompanied with a loss of memory [11] and a decay of correlation in a short time, auto-correlation function $C(\tau)$ of chaos have a peak at the origin $\tau=0$ and decrease quickly [10]. Chaotic behavior and white noise are similar in appearance, but difference between chaos and white noise may appear in the power spectral density : continuous broadband and damping spectrum for chaos vs. continuous and flat spectrum for white noise [10, 12].

§ 4. Numerical Simulations of Several Equations and Discussions

In this section the numerical simulations of several equations are shown, representing trajectory of pattern translation and the Fourier components of their pattern, a time series of them and its Fourier mode, and their phase portraits in the time difference and the space difference which are figures plotted f_n^t in horizontal axis and its time derivative or space derivative in vertical axis.

As initial conditions we take the pattern (Fig.1(a)) which has three frequency modes (Fig.1(b)), and translate this pattern obeying dynamical systems. These systems are presented in differential equations and disceretized by simple difference method, except in the case of Eq. (4) for Volterra model Eq. (3). Other equations are follows :

Volterra model plus quadratic term

$$\frac{df_n}{dt}=(f_{n+1}-f_{n-1})f_n+f_n^2 \tag{12}$$

Toda lattice model,

$$\frac{\partial^2 f_n}{\partial t^2}=\exp(f_{n-1}-f_n)-\exp(f_n-f_{n+1}) \tag{13}$$

KdV model

$$\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial x}+\frac{\partial^3 f}{\partial x^3}=0 \tag{14}$$

and convection equation as linear reference system

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \tag{15}$$

Trajectories of the pattern translation (a) and its Fourier modes in wave-number space (b) for the above-mentioned equations are shown in Fig. 2-7(a),(b) respectively. Regarding the pattern translations as one time series, we transform them to the Fourier space and calculate auto-correlation functions of the series. However there seems problem to calculate auto-correlation function and the power spectral density for finite periodic data like our case. So we show that time series (c), Fourier modes of the time series (d) and phase portraits in the time difference (e) and the space difference (f) in Fig. 2-7(c),(d),(e) and (f) for the above cases respectively. From these figures we see that in the real space it is difficult to distinguish whether the translation of pattern is stable or not. In the Fourier space, however, the degree of disoder of the translation of pattern becomes clear. There remains the possibility of picking up multiplier modes as in the case of Fig.4. They are subjects that Fourier transform by FFT gives correct Fourier modes or not for finite data and that statistical methods like as the auto-correlation function and the power spectral density are suitable or not for finite periodic data and unstable phenomena. Figures of phase portraits shows symmetrical property of time and space variables, and their symmetricity depend much more on the rank of derivative than on integrability. Fourier transform is useful tool, but have to be treated carefully. To classify chaotical behaviors and integrability or to know transition from integrability to chaotical

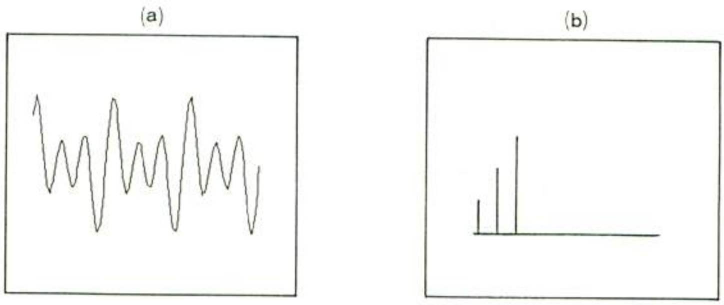

Fig.1 (a) Initial pattern and (b) its Fourier mode.

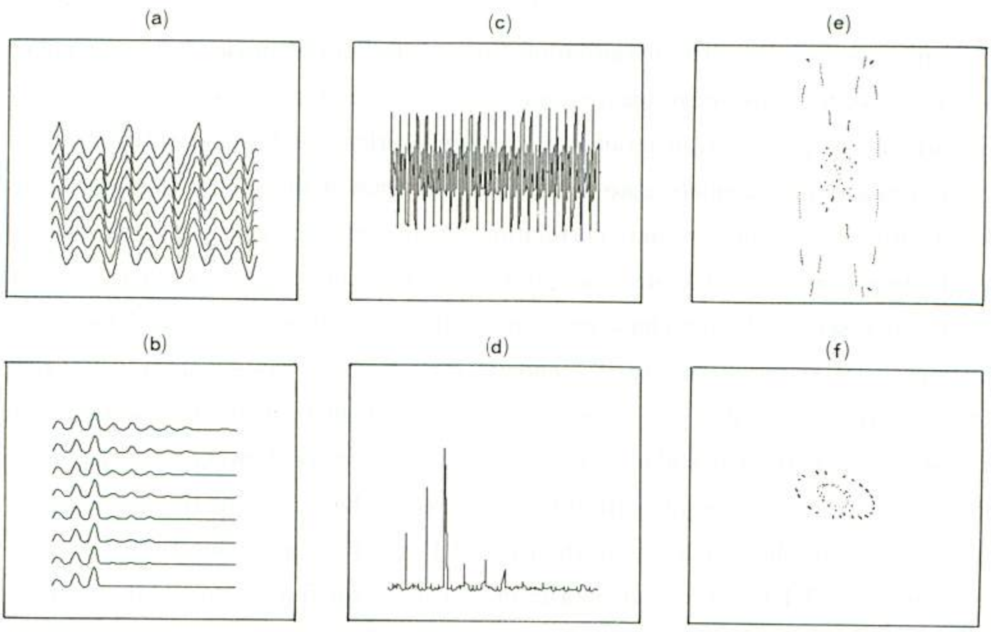

Fig.2 (a) Trajectories, (b) Fourier mode of (a), (c) the time series, (d) Fourier mode of (c), (e) phase portrait of the time difference and (f) phase portrait of the space difference for Eq. (4).

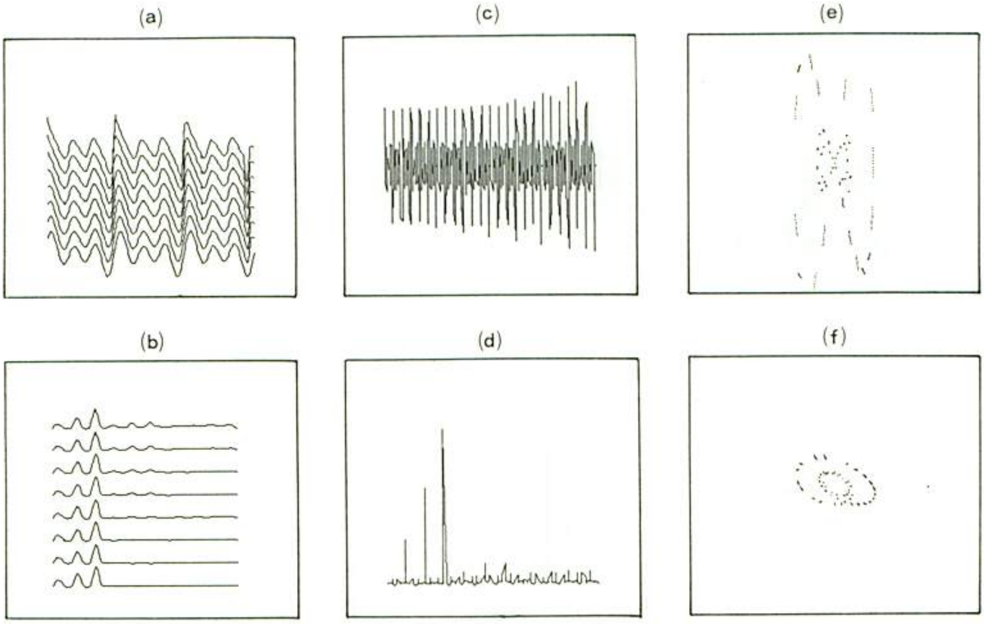

Fig.3 Same as in Fig.2 but for Eq.(5).

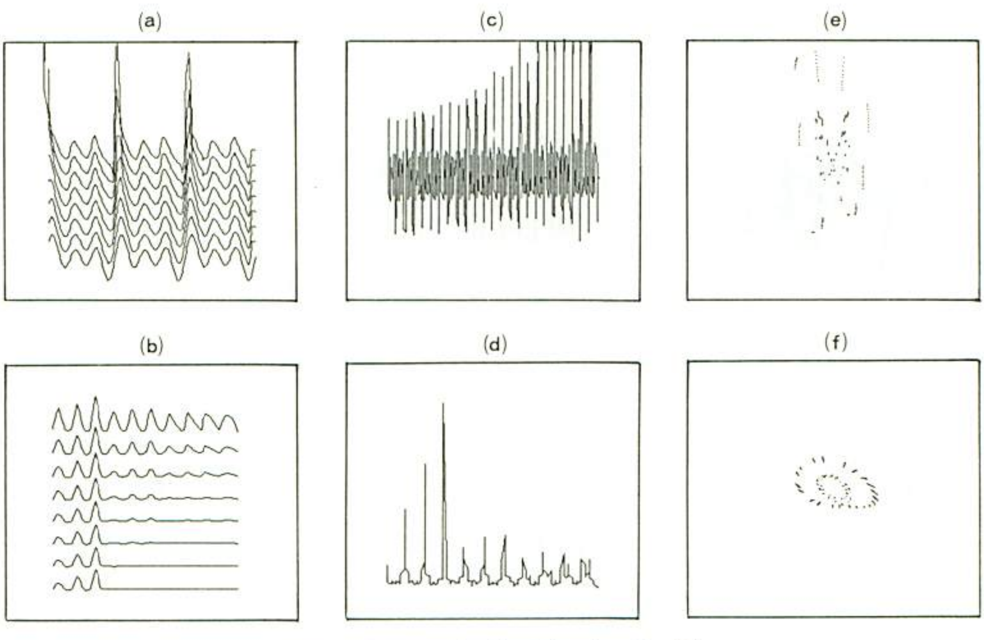

Fig.4 Same as in Fig.2 but for Eq. (12).

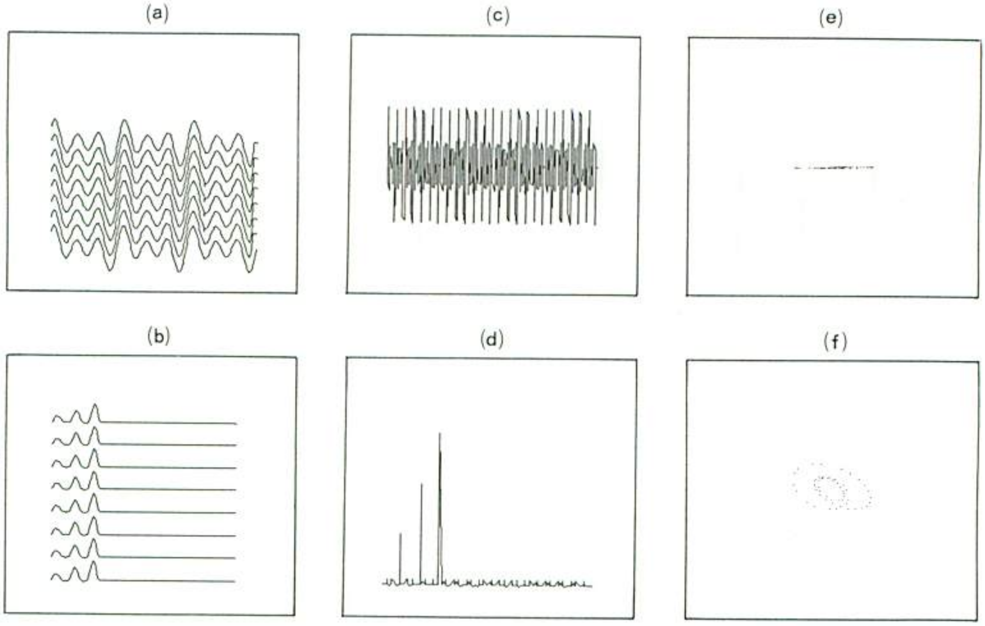

Fig.5 Same as in Fig.2 but for Eq. (13).

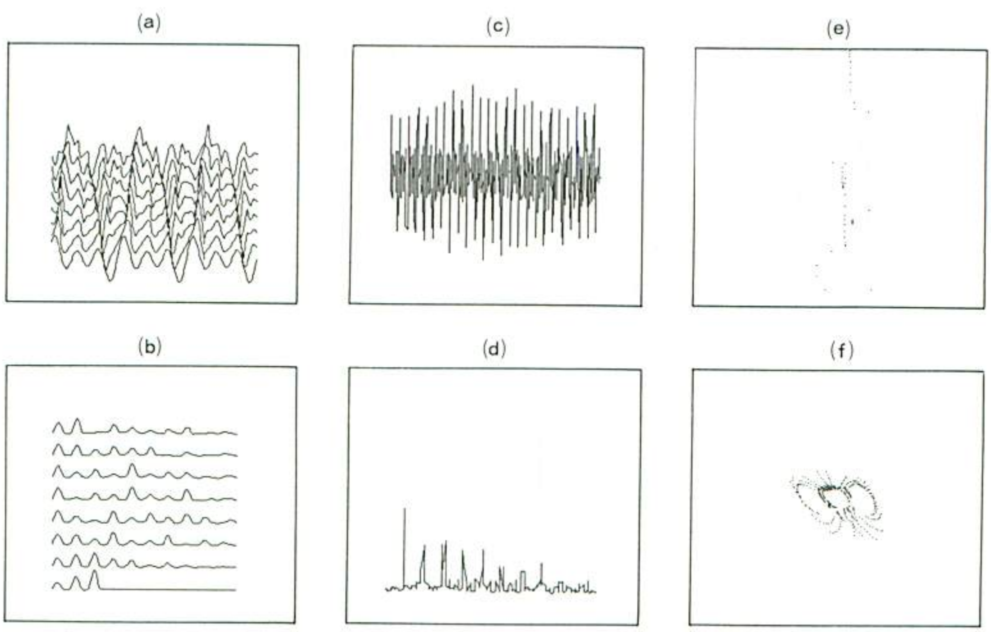

Fig.6 Same as in Fig.2 but for Eq. (14).

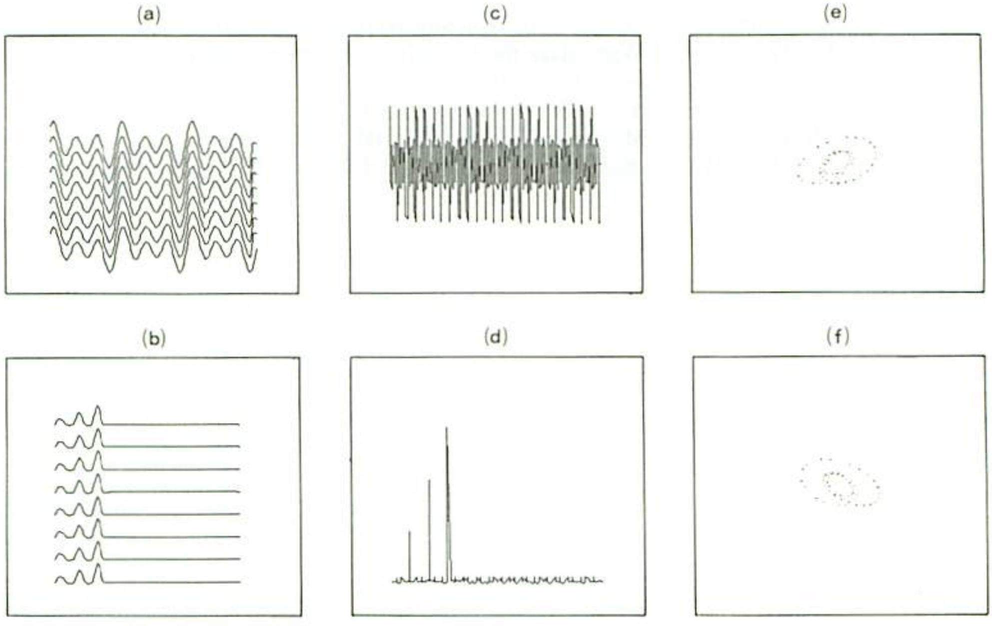

Fig.7 Same as in Fig.2 but for Eq. (15).

behavior we need various tools of quantitative treatments.

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